21/11/2024

TEOREMA (BANACH ALAOGLU, COMPATTEZZA SEQ.)

E BANACH SEPARABILE.

$u_n \in E^*$, $\|u_n\|_{E^*} \leq C \implies \exists\, u_{n_k} \xrightarrow[*]{} u$ IN $\sigma(E^*, E)$,

CIOÈ $u_{n_k}(x) \xrightarrow[k]{} u(x) \quad \forall x \in E$.

DIM. x_j DENSO IN E $\quad |u_n(x_j)| \leq \|u_n\|_{E^*} \cdot \|x_j\| \leq C\|x_j\| \quad \forall n$

$\exists\, u_{n_k^1}$ SOTTOSUCC. T.C. $u_{n_k^1}(x_1) \xrightarrow[k]{} u(x_1) \in \mathbb{R}$

$\exists\, u_{n_k^2}$ SOTTOSUCC. DI $u_{n_k^1}$ T.C. $u_{n_k^2}(x_2) \xrightarrow[k]{} u(x_2)$

$\vdots \qquad \vdots$

$\exists\, u_{n_k^j}$ " " $u_{n_k^{j-1}}$ T.C. $u_{n_k^j}(x_j) \xrightarrow[k]{} u(x_j)$

CONSIDERO LA SOTTOSUCC. $\tilde{u}_k = u_{n_k^k}$

OSSERVIANO CHE $\tilde{u}_k(x_j) \xrightarrow[k]{} u(x_j) \quad \forall j$

$$|u(x_j) - u(x_\ell)| = \lim_k |u_k(x_j) - u_k(x_\ell)| \leq C \|x_j - x_\ell\|_E$$

u È C-Lipschitz $\Rightarrow \exists! \, \tilde{u} : E \to \mathbb{R}$ ESTENSIONE C-LIP. DI u.

INOLTRE $\tilde{u}(x) = \lim_k u_k(x) \quad \forall x \in E \quad (*)$

NE SEGUE CHE $\tilde{u} \in E^*$ E $u_k \xrightarrow[*]{} \tilde{u}$ IN $\sigma(E^*, E)$.

VERIFICHIANO $(*)$:

$$|\tilde{u}(x) - u_k(x)| \leq |\tilde{u}(x) - \tilde{u}(x_j)| + |u(x_j) - u_k(x_j)| + |u_k(x_j) - u_k(x)|$$

$$\leq 2C\|x - x_j\|_E + |u(x_j) - u_k(x_j)|$$

$\forall \varepsilon \; \exists j$ T.C. $\|x - x_j\| \leq \varepsilon$ ED $\exists K_\varepsilon$ T.C. $|u(x_j) - u_k(x_j)| \leq \varepsilon \quad \forall K > K_\varepsilon$

$$\Rightarrow |\tilde{u}(x) - u_k(x)| \leq (2C+1)\varepsilon \quad \forall K > K_\varepsilon \Rightarrow (*).$$

OSS: E SEPARABILE $\Rightarrow \left(\overline{B}_R^{E^*}, \sigma(E^*, E) \right)$ È METRIZZABILE

$\overline{B}_R^{E^*} = \{ u \in E^* : \|u\| \leq R \}$

$\Rightarrow$ LA COMPATTEZZA DI $\overline{B}_R^{E^*}$ È EQ. ALLA COMPATTEZZA SEQ.

OSS (BANACH ALAOGLU):

IN GENERALE, CIOÈ $\forall E$, POSSO DIRE CHE

$\overline{B}_R^{E^*}$ È COMPATTO PER $\sigma(E^*, E)$.

COR: •) ℓ_p, $L^p(\Omega)$ È RIFLESSIVO E SEPARABILE $\forall\, 1 < p < +\infty$

$$\ell_p^* = \ell_q, \quad L^p(\Omega)^* = L^q(\Omega) \qquad q = \frac{p}{p-1}$$

$\Rightarrow$ LE PALLE CHIUSE DI ℓ_p, L^p SONO SEQ. DEB. COMPATTE

(LA TOP. DEBOLE COINCIDE CON LA DEBOLE *)

•) $\ell_\infty = \ell_1^*$, $L^\infty = L^{1^*}$, ℓ_1 e L^1 SONO SEPARABILI

$\Rightarrow$ LE PALLE CHIUSE DI ℓ_∞, L^∞ SONO SEQ. DEB.* COMPATTE

•) LE PALLE CHIUSE DI ℓ_1, L^1 NON SONO SEQ. DEB. COMPATTE

ES (ℓ_1): $e_n = (0, \dots, \underset{\substack{\uparrow \\ n^a \text{ POSIZ.}}}{1}, 0, \dots)$, $\nexists\, x \in \ell_1$ T.C. $e_n(y) = y_n \underset{n}{\rightarrow} \sum_k x_k y_k$ $\forall y \in \ell_\infty$

$y = e_k \in \ell_\infty \Rightarrow \forall n > k \quad e_n(y) = 0 = x_k \Rightarrow x = 0$ MA $y = (1, \dots 1, \dots)$
$\forall n \quad e_n(y) = 1 = \sum x_k = 0$ ASSURDO

ESISTENZA IN $W^{1,p}(I)$

$1 < p < +\infty$

$u_n \in W^{1,p}$, $\|u_n\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p} \leq C \implies$ (B.A. sep.)

$\exists\, u, v \in L^p$ T.C. $u_{n_k} \underset{k}{\rightharpoonup} u$ E $u'_{n_k} \underset{k}{\rightharpoonup} v$ DEB. IN L^p

$$\int_I v\varphi = \lim_k \int_I u'_{n_k}\varphi = -\lim_k \int_I u_{n_k}\varphi' \doteq -\int_I u\varphi' \quad \forall \varphi \in C_c^1(I)$$

$\implies u \in W^{1,p}$ E $v = u'$.

SCRIVIAMO $u_{n_k} \underset{k}{\rightharpoonup} u$ DEB. IN $W^{1,p}$

PROP: $u_n \to u$ IN $W^{1,p} \implies u_n \to u$ UNIF. IN PART. $u_n \underset{n}{\to} u$ IN L^p

DIM $\|u_n\|_{W^{1,p}} \leq C \implies \|u_n\|_{L^\infty} \leq C$ E u_n EQUI-HÖLDER $\implies$ EQUICONTINUE

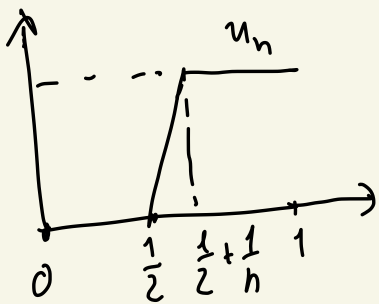
$$|u_n(x)-u_n(y)| = \left|\int_x^y u_n'\right| \le \int_x^y |u_n'|\cdot 1 \underset{\text{HÖLDER}}{\le} \|u_n'\|_{L^p}\left(\int_x^y 1^q\right)^{\frac{1}{q}}$$

$$q = \frac{p}{p-1} \qquad\qquad \le C\,|x-y|^{\frac{1}{q}}$$

LA TESI SEGUE DA ASCOLI-ARZELÀ.

OSS: $\exists\, u_n \in W^{1,1}$, $\|u_n\| \le C$ T.C. $\lim_n u_n(x) = u(x)$ PER q.o. x

MA $u \notin W^{1,1}$



$$u_n(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ nx - \frac{n}{2} & \frac{1}{2} \le x \le \frac{1}{2}+\frac{1}{n} \\ 1 & \frac{1}{2}+\frac{1}{n} < x \le 1 \end{cases}$$

$$u = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ 1 & \frac{1}{2} < x \le 1 \end{cases} \notin W^{1,1}$$

TEO (ESISTENZA DI MIN. IN $W^{1,p}$)

$1 < p < +\infty$ $\qquad \mathcal{L}(u) = \int_I L(x,u,u')\,dx$

L VERIFICA:

1 L DI CARATHEODORY

2 $z \to L(x,y,z)$ CONVESSA $\forall x, y$

3 $L(x,y,z) \geq \alpha |z|^p + \beta \qquad \alpha > 0$ (COERCIVITÀ)

$\Rightarrow \exists \min\limits_{A} \mathcal{L}$ DOVE $A = \{ u \in W^{1,p} : u - u_0 \in W_0^{1,p}$ CON $u_0 \in W^{1,p}$ FISSATA $\}$ T.C. $\mathcal{L}(u_0) < +\infty$

SIGNIFICA "A ESTREMI FISSATI"

SE $(y,z) \to L(x,y,z)$ È STR. CONVESSA $\tilde{\forall} x \Rightarrow$

IL MINIMO È UNICO

DIM: SIA u_n SUCC. MINIMIZZANTE, CIOÈ $\mathcal{L}(u_n) \to \inf_{A} \mathcal{L} \leq \mathcal{L}(u_0) = C$

$$\overset{③}{\Longrightarrow} \int_I \alpha |u_n'|^p + \beta |I| \leq \mathcal{L}(u_n) \leq C$$

$$\Rightarrow \int_I |u_n'|^p \leq C_1 \Rightarrow \int |(u_n - u_0)'|^p \leq C + \|u_0'\|_{L^p} = C_2$$

$$\underset{\text{POINCARÉ}}{\Longrightarrow} \|u_n - u_0\|_{L^p} \leq C_3 \Rightarrow \|u_n\|_{L^p} \leq C_4 \Rightarrow \|u_n\|_{W^{1,p}} \leq C_5$$

PER BANACH-ALAOGLU $\exists\, u \in A$ E n_k T.C.

$$u_{n_k} \xrightarrow[k]{} u \text{ IN } W^{1,p}.$$

MOSTRIAMO CHE $\mathcal{L}(u) \leq \liminf_{k} \mathcal{L}(u_n) = \inf_{A} \mathcal{L}$

RAFFORZIAMO LE IPOTESI:

(5) $\tilde{\forall} x \quad (y,z) \to L(x,y,z)$ È CONVESSA, C^1 E SI HA

$$|L_y(x,y,z)| + |L_z(x,y,z)| \leq C\left(1 + |y|^{p-1} + |z|^{p-1}\right)$$

$$\mathcal{L}(u_n) = \int_I L(x,u_n,u_n') \geq \int \underbrace{L(x,u,u')}_{\mathcal{L}(u)} + \overset{L^q}{L_y} \cdot \underbrace{(u_n - u)}_{\downarrow n \ 0} + \overset{L^q}{L_z} \underbrace{(u_n' - u')}_{\downarrow n \ 0}$$

SE $(y,z) \to L(x,y,z)$ È STRETT. CONVESSA $\Rightarrow$ $u \to \mathcal{L}(u)$ È STRETT. CONVESSA

$$\mathcal{L}\left(\frac{u_1+u_2}{2}\right) < \frac{\mathcal{L}(u_1) + \mathcal{L}(u_2)}{2}$$

$\Rightarrow$ IL MINIMO È UNICO.